ON THE REGULARIZATION OF SENSE AND SPACE-RIP IN PARALLEL MR IMAGING

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ABSTRACT

Parallel imaging methods provide accelerated multiple coil MR image acquisitions via reconstruction of sub-sampled *k*-space data. Currently, analytic comparison between different reconstruction approaches has been hampered by use of different phase encoding paradigms and regularization approaches, historically unique to each method. We present an analysis of the Space-RIP image reconstruction problem that demonstrates the ability to recast the problem in a decoupled form when uniform down-sampling is employed. We show that this decoupled problem is equivalent to the SENSE image reconstruction approach. This approach enables a clear analytic comparison between SENSE and Space-RIP, and we demonstrate the effect of different regularization approaches on image formation in low coil sensitivity regions.

1. INTRODUCTION

In pursuit of achieving reduced MR image acquisition time, a number of image reconstruction methods have been proposed. Parallel imaging methods distribute the data acquisition burden across multiple receiver coils and then subsample during data acquisition to reduce total imaging time. The strength of these reconstruction methods is the ability to suppress aliasing artifacts in the reconstructed image.

The parallel MR imaging process is a classic inverse problem, in that an image of the encoded tissue is generated by indirect means. Image reconstruction is highly sensitive to small perturbations, both in the received data and in the estimates of the coil sensitivity maps. Regularization techniques are widely deployed to limit the impact of such noise on the reconstructed image quality [1, 2]

Previous comparisons between parallel imaging methods have focused primarily on SMASH and SENSE [3, 4, 5]. However, there is still some debate as to the how best to differentiate between all methods. And to our knowledge, no comparison of regularization methods has been presented.

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Here we present an analysis that plainly illustrates the connection between SENSE and Space-RIP. This analysis allows these separate methods to be described using a common analytical framework, which enables comparison of the regularization approaches particular to each method.

2. BACKGROUND

2.1. Brief Review of Regularization

The image reconstruction problem posed in parallel imaging is a classic ill-posed inverse problem. This is a direct consequence of attempting to determine system parameters, in this case the underlying spin distribution in the field of view (FOV), by indirect means. When such a system is solved computationally, small disturbances in the acquired data tend to amplify and corrupt the solution with noise. System regularization is often employed to limit these effects.

For the regularization review below, we present a generic linear least squares problem

$$Ax = b, \quad \text{or equivalently,} \quad \min_x \; \|Ax - b\|_2^2, \qquad (1)$$

and assume that the singular value decomposition of a matrix A is known, i.e., $A = U\Sigma V^H = \sum_{i=1}^{n} u_i \sigma_i v_i^H$. One method to limit noise amplification in this inverse problems is to compute the *truncated SVD solution* [6]. This approach finds the best rank k approximation to the system matrix, $A_k = \sum_{i=1}^{k} u_i \sigma_i v_i^H$, by truncating the singular values below a given threshold, ε. This prevents the smallest singular values, which are assumed to be associated with noise in the acquired data, from impacting the system solution. Ideally, the choice for k occurs at the index i where an obvious gap occurs in the sequence of singular values, i.e., $\sigma_k > \varepsilon \gg \sigma_{k+1} > \sigma_{k+2} \cdots > \sigma_n$.

An alternative approach is to employ a *damped least squares* (DLS), or Tikhonov, solution [6]. This approach considers an alternate problem

$$\min_x \; \|Ax - b\|_2^2 + \tau^2 \|Dx\|_2^2, \qquad (2)$$

to constrain the computed solution x. Selecting a diagonal weighting matrix D allows fine control in limiting the power across the solution. When the regularization parameter is positive, $\tau > 0$, this problem has full column rank and a unique solution [6]. If the weighting matrix is equal to the identity matrix, $D = I$, then the solution to 2 can be expressed in terms of the SVD as (errata: c_i = u_i^H b)

$$x(\tau) = \sum_{i=1}^{n} \frac{c_i f_i}{\sigma_i} v_i, \quad f_i = \frac{\sigma_i^2}{\sigma_i^2 + \tau^2}. \tag{3}$$

The quantities f_i are called *filter factors* [6], and couple Tikhonov regularization to the truncated SVD approach. Notice that if $\tau \ll \sigma_i$, then $f_i \approx 1$. Similarly, if $\tau \gg \sigma_i$, then $f_i \ll 1$. This establishes the fact that for $\tau = \sum_{i=k+1}^{n} \sigma_i$, the truncated SVD and DLS solution to Eq. (1) are approximately the same, particularly for those cases where a large gap between the singular values σ_k and σ_{k+1} exists [6].

2.2. Brief Review of Parallel Imaging

Parallel MR imaging methods aim to reconstruct an image of the excited FOV spin distribution from down-sampled data acquired using multiple coils. The signal acquired in each coil, l, can be described by

$$s_l(\mathbf{k}) = \int_V W_l(\mathbf{r}) \rho(\mathbf{r}) e^{j2\pi \mathbf{k}\cdot\mathbf{r}} d\mathbf{r}. \tag{4}$$

where $\rho(\mathbf{r})$ is the excited spin density function throughout the volume V weighted by the various imaging factors (e.g., relaxation terms, hardware characteristics etc.), $\mathbf{r}$ is the spatial position in the FOV, $W_l(\mathbf{r})$ is the coil sensitivity at $\mathbf{r}$, and $\mathbf{k}$ is a reciprocal spatial term corresponding to the gradients employed during data acquisition, $\mathbf{k} = \gamma \int_0^t \mathbf{G}(t) dt$.

This signal representation is typically discretized to solve the inverse problem computationally. As originally presented, Cartesian SENSE [7] imposes a uniform down-sampling pattern. Based on the subsequent aliasing pattern, one can then construct a small system for each spatial-domain pixel in the acquired data reference frame. Solving this small system gives un-aliased spatial-domain pixels. This process is then repeated for each pixel in the FOV. In Space-RIP [8], an FFT is performed along one dimension to consider each column of the FOV in turn. This results in a large linear system

$$\bar{s} = \mathbf{P}\bar{\rho} \tag{5}$$

where the vector $\bar{s}$ contains the output data from all coils for a single column, $\bar{\rho}$ is the desired FOV, and $\mathbf{P}$ is a system matrix constructed from knowledge of the phase-encoding pattern used and the coil sensitivity estimates. Generalized SENSE [9] can also be described by Eq. 5. As with any linear system, one can alternatively frame the reconstruction as a minimization problem

$$\min_{\bar{\rho}} \|\bar{s} - \mathbf{P}\bar{\rho}\|_2, \tag{6}$$

with the possibility of adding regularization terms as in Eq. (2). It is from this minimization perspective that we examine regularization in parallel MR imaging.

2.3. A closer look at Space-RIP

We examine here the analytic details of Space-RIP to enable a comparison with SENSE. This choice is guided by two features: the problem can be described in 1D, and the particular phase encode sampling pattern employed does not impact the implementation in any way.

For the case of 2D imaging with phase-encoding along the k_y direction, Kyriakos, et. al., [8], present the following equation,

$$s_l(k_y, x) = \sum_{n=1}^{N} \rho(x,n) W_l(x,n) e^{jk_y n\tau}, \tag{7}$$

as a discrete version of Eq. 4 after the application of a DFT along k_x. One can recast this equation in compact form as

$$s_l(:,x) = P^H \operatorname{diag}\{W_l(x,:)\} \rho(x,:)$$

where all of the acquired data points associated with a particular column x have been collected into a single vector of length M, corresponding to the number of phase encodes employed. The elements of the M-by-N matrix P^H correspond to the exponential term in (7), $P(n, k_y) = e^{-jk_y n\tau}$, where each row of P^H corresponds to a particular phase encode k_y. We note here that the columns of P (and rows of P^H) correspond to columns of the discrete Fourier transform operator, and are mutually orthogonal.

Recognizing that the underlying spin density terms $\rho(x,:)$ are common in each of the coils, the Space-RIP linear system for column x can be written as

$$\begin{bmatrix} s_1(:,x) \\ s_2(:,x) \\ \vdots \\ s_L(:,x) \end{bmatrix} = \begin{bmatrix} P^H \operatorname{diag}\{W_1(:,x)\} \\ P^H \operatorname{diag}\{W_2(:,x)\} \\ \vdots \\ P^H \operatorname{diag}\{W_L(:,x)\} \end{bmatrix} \rho(x,:) \tag{8}$$

2.4. Common ground between parallel MR methods

The common ground between parallel imaging methods can be demonstrated by solving the minimization problem in Eq. (6), using the system matrix given in Eq. (8). Differentiating Eq. (6) with respect to $\bar{\rho}$ yields the normal equations

$$\mathbf{P}^H \bar{s} = \mathbf{P}^H \mathbf{P} \bar{\rho}. \tag{9}$$

This representation is useful in that the system matrix is conjugate-symmetric, a prerequisite if one wishes to use the conjugate gradient (CG) algorithm [10] to solve the linear system, as in [9].

In the case of Space-RIP, closer inspection of Eq. (9) reveals some interesting consequences. In particular, using the system matrix in Eq. (8) for $\mathbf{P}$, one finds

$$\mathbf{P}^H\mathbf{P} = \left[\left(PP^H\right) \circ \left(\sum_{l=1}^{L} W_l(:,x) W_l(:,x)^H \right) \right] \quad (10)$$

where $\circ$ represents an element-by-element matrix product. In this form, the interaction between the selected phase encodes and the coil sensitivities is readily apparent.

PP^H is effectively an "aliasing" operator. To see this, recall that the columns of P are equivalently columns of a Fourier transform matrix, F, and are mutually orthogonal. Downsampling uniformly by f results in

$$P(fm,n) = \left\{ e^{-j2\pi fmn/N} \right\}, \;\; 0 \leq m,n \leq N-1.$$

With a change of variables, one can rewrite this matrix as

$$P(o,n) = \left\{ e^{-j2\pi on/(N/f)} \right\}, \quad \begin{array}{c} 0 \leq n \leq N-1, \\ 0 \leq o \leq (N/f-1) \end{array} .$$

or more simply, $P = [\; F_{N/f} \; F_{N/f} \; \cdots \;]$, with the submatrix repeated f times. In descriptive terms, uniform down sampling along the phase encode direction produces a matrix P that contains multiple copies of a small Fourier transform operator. Thus, in the case of down-sampling by $f = 2$, the outer product of PP^H results in

$$PP^H = N/2 \begin{bmatrix} I_{N/2} & I_{N/2} \\ I_{N/2} & I_{N/2} \end{bmatrix}. \quad (11)$$

which is clearly an aliasing operator.

In (10), this aliasing operator *masks* the coil sensitivity information that interacts with the spin density data, ρ, which is ultimately seen in the output data. For the $f = 2$ example above, the system matrix $[(PP^H) \circ (\sum_{l=1}^{L} W_l(:,x) W_l(:,x)^H)]$ can be permuted into a block diagonal matrix, with the j^{th} block as given in Eq. (12). The elements of each block can be constructed from an inner-product, $B_{jj} = U_j^H U_j$, where

$$U_j = \begin{bmatrix} W_1(j,x) & W_1(j+N/2,x) \\ W_2(j,x) & W_2(j+N/2,x) \\ & \vdots \\ W_L(j,x) & W_L(j+N/2,x) \end{bmatrix}. \quad (13)$$

This matrix U_j is identical to the matrix description in Cartesian SENSE [7], which solves the small system

$$\begin{bmatrix} s_1(j,x) \\ s_2(j,x) \\ \vdots \\ s_L(j,x) \end{bmatrix} = \begin{bmatrix} W_1(j,x) & W_1(j+N/2,x) \\ W_2(j,x) & W_2(j+N/2,x) \\ & \vdots \\ W_L(j,x) & W_L(j+N/2,x) \end{bmatrix} \tilde{\rho} \quad (14)$$

with $\tilde{\rho} = \left[\; \rho(j,x) \quad \rho(j+N/2,x) \;\right]^T$, for each pixel location in the spatially aliased image.

From this result we draw the following conclusion: with uniform down-sampling, the normal equations for SENSE and Space-RIP are analytically equivalent. Each solves the multiple coil signal equation in the same way. The primary difference between the two is that SENSE repeatedly solves a number of small linear systems, where-as Space-RIP collects all of the small systems for a single column and finds the solution simultaneously.

This analysis explicitly states what has been the conventional wisdom for some time. Specifically, with uniform downsampling, SENSE is an efficient solution to the image reconstruction problem. However, the limited number of parameters in the linear system requires the deployment of Tikhonov regularization. In contrast, Space-RIP collects the aliasing interaction between a number of FOV pixels into a single system. The potential advantages of this later approach include both the freedom to choose acquisition phase encodes — allowing noise-reduction averaging through coupling across multiple pixels — and greater freedom in regularization.

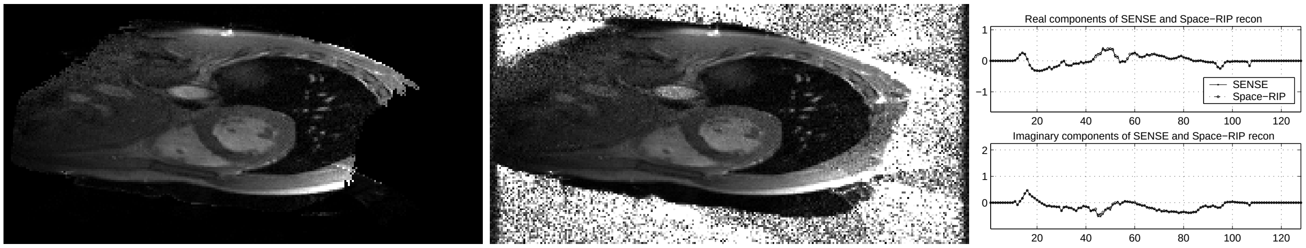
3. REGULARIZATION IN PARALLEL IMAGING

With the relationship between SENSE and Space-RIP now clearly illustrated, one can compare regularization methods that were previously used independently in each method. The use of filter factors allows the Tikhonov regularization parameter typically used in SENSE to map directly to the truncated SVD threshold typically used in Space-RIP. The example below shows SENSE and Space-RIP images for data from a 4-coil, $f = 2$, cardiac sequence. The coil sensitivities were estimated from a water phantom acquisition prior to the volunteer being placed within the magnet.

The regularization parameter was set based on the Space-RIP system for column 101, and then fixed for the remaining columns in the rest of image. This approach was chosen because it most clearly demonstrates the effect of each regularization approach on the image reconstruction. In practice one could reset the regularization parameter for each inverted system, although the optimal scheme for regularization remains an open question.

The tSVD parameter was chosen at that point in the singular value distribution where a large gap in the values exist, $\varepsilon = 9.52 \times 10^2$, illustrated in Figure 2. The DLS parameter was then selected based on the discarded singular values. The relationship between the SENSE and Space-RIP systems are related through the normal equations, thus the DLS regularization parameter is set as $\alpha = \sqrt{\sum_{i=k+1}^{n} \sigma_i^2} = 1.063 \times 10^3$.

$$B_{jj} = \begin{bmatrix} \sum_{l=1}^{L} W_l^*(j,x) W_l(j,x) & \sum_{l=1}^{L} W_l^*(j,x) W_l(j+N/2,x) \\ \sum_{l=1}^{L} W_l^*(j+N/2,x) W_l(j,x) & \sum_{l=1}^{L} W_l^*(j+N/2,x) W_l(j+N/2,x) \end{bmatrix}. \quad (12)$$



(a) (b) (c)

Fig. 1. Comparison of (a) tSVD Space-RIP and (b) damped-least-squares SENSE reconstructions at identical regularization and windowing levels. (c) shows the complex valued data in the anatomical region for column 101 of each reconstruction.

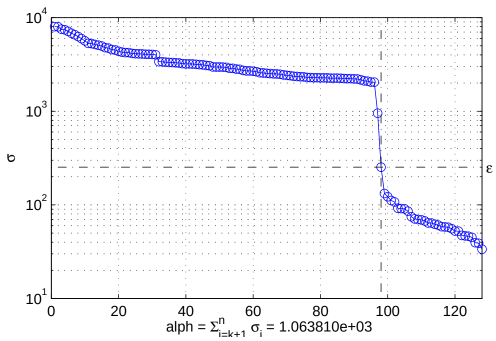


Fig. 2. Singular value distribution for column 101, with the truncated SVD threshold, ε, shown

Fig. 1 shows the reconstructions for both a tSVD Space-RIP reconstruction, and a DLS SENSE reconstruction. As can be seen in Fig. 1(c), in the region where there is significant coil sensitivity information, the anatomical reconstructions are practically identical.

The main difference between the two methods of regularization appears in regions with very low coil sensitivity. The tSVD method thresholds these regions to zero, which leads to very clean images, at the possible expense of anatomical feature loss. In contrast, the DLS method introduces conditioning into the system matrix. Thus, the regions outside the coil sensitivity may still be reconstructed but with very noisy signal. This illustrates the importance of reconstruction masking, allowing only those pixels covered by the sensitivity estimates in final reconstructed image.

4. CONCLUDING REMARKS

Through a normal equations framework, we have demonstrated a mechanism to analytically compare two parallel MR imaging methods, SENSE and Space-RIP. Based on this analysis, it was shown that practically identical images can be formed through uniform downsampling and careful control of the regularization parameters. The possible extension of this result to irregular sampling patterns is currently under study.

Our results emphasize that the choice of regularization method can be made independently of the parallel MR reconstruction method used. In the parallel imaging process, the aliasing artifacts and system ill-conditioning come from distinct sources, including the sub-sampling pattern and low non-orthogonal coil sensitivity respectively. The mechanism in which they are coupled in the imaging process, and the appropriate regularization and optimal reconstruction system to decouple them is an open question and subject to more study.

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