Invariant Crease Lines for Topological and Structural Analysis of Tensor Fields

Xavier Tricoche\textsuperscript{1}, Gordon Kindlmann\textsuperscript{2}, Carl-Fredrik Westin\textsuperscript{2}

\begin{itemize}
\item \textsuperscript{1} Computer Science Dept., Purdue University
\item \textsuperscript{2} LMI, Brigham and Women’s Hospital, Harvard Medical School
\end{itemize}
Tensor Fields in Visualization

• Strain / stress in continuum mechanics
  solid mechanics

fluid dynamics
Tensor Fields in Visualization

\[ A_i(b, g) = A_0 e^{-b g_i^T D g_i} \]

(Basser 1994)

\[ A_0, A_i \]

\[ D_{xx}, D_{xy}, D_{xz}, D_{yy}, D_{yz}, D_{zz} \]

fractional anisotropy (FA)
Tensor Field Structure

Double Point Load

Diffusion Tensor Brain
Outline

• Context and motivation
• Theory and related work
• Algorithm
• Results
Road Map

Tensor Math
- Eigenvector
- Eigenvalue
- Invariants

Tensor Vis
- Topological approach
- Our Method

Vector Vis
- Invariants
- Feature extraction

Scalar Image Analysis
- Creases
Tensor Field Basics

$T = RDR^{-1}$

Eigenvalues = shape

Eigenvectors = orientation

Eigenvector fields
Road Map

**Tensor Math**
- Eigenvector
- Eigenvalue
- Invariants

**Tensor Vis**
- Topological approach

**Vector Vis**
- Invariants
- Feature extraction

**Our Method**

**Scalar Image Analysis**
- Creases
Topological Methods

• Application to tensor fields
  (Delmarcelle & Hesselink, '94)

• topology of eigenvector fields

• degenerate points: 2, 3 eigenvalues equal

\[
\begin{align*}
\lambda_1 &= \lambda_2 \\
\lambda_2 &= \lambda_3 \\
\lambda_1 &= \lambda_2 = \lambda_3
\end{align*}
\]
Topological Methods

• Application to tensor fields
  (Delmarcelle & Hesselink, ’94)

• topology of eigenvector fields

• degenerate points: 2, 3 eigenvalues equal

Zheng & Pang, ’04
Tensor Field Topology

• Extraction of degenerate lines

• Zheng & Pang, ‘04, ’05

• local criterion: \[ D_3(T) = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2 = 0 \]

\[
\begin{align*}
x(T) &= T_{00}(T_{11}^2 - T_{12}^2) + T_{00}(T_{01}^2 - T_{02}^2) + T_{11}(T_{22}^2 - T_{00}^2) + T_{11}(T_{12}^2 - T_{01}^2) + T_{22}(T_{00}^2 - T_{11}^2) + T_{22}(T_{02}^2 - T_{12}^2) \\
y_1(T) &= T_{12}(2(T_{12}^2 - T_{00}^2) - (T_{02}^2 + T_{01}^2) + 2(T_{11}T_{00} + T_{22}T_{00} - T_{11}T_{12})) + T_{01}T_{02}(2T_{00} - T_{22} - T_{11}) \\
y_2(T) &= T_{02}(2(T_{02}^2 - T_{11}^2) + (T_{01}^2 + T_{12}^2) + 2(T_{22}T_{11} + T_{00}T_{11}) \\
D_3(T) &= x(T)^2 + y_1(T)^2 + y_2(T)^2 + y_3(T)^2 + 15f_1(T)^2 + 15f_2(T)^2 + 15f_3(T)^2 \\
f_2(T) &= T_{02}(T_{01}^2 - T_{12}^2) + T_{12}T_{01}(T_{22} - T_{00}) \\
f_3(T) &= T_{01}(T_{12}^2 - T_{02}^2) + T_{02}T_{12}(T_{00} - T_{11})
\]
Tensor Field Topology

• Shortcomings for DTI data

 tractography
  degenerate lines (planar type)
  degenerate lines (linear type)

from Schultz et al., ‘07
Ridges and Valleys

- Used in computer vision to find skeleton features in scalar images

Morse, ‘94
Crease feature definition (Eberly 1994)

- Constrained extremum
- Gradient $\mathbf{g}$
- Hessian eigensystem $\mathbf{e}_i, \lambda_i$
- Crease: $\mathbf{g}$ orthogonal to one or more $\mathbf{e}_i$
- Eigenvalue gives strength

Ridge line: $\mathbf{g} \cdot \mathbf{e}_3 = \mathbf{g} \cdot \mathbf{e}_2 = 0; \lambda_3, \lambda_2 < \text{thresh}$

Valley line: $\mathbf{g} \cdot \mathbf{e}_2 = \mathbf{g} \cdot \mathbf{e}_1 = 0; \lambda_1, \lambda_2 > \text{thresh}$
Road Map

Tensor Math
  Eigenvector
  Eigenvalue
  Invariants

Tensor Vis
  Topological approach

Our Method

Vector Vis
  Invariants
  Feature extraction

Scalar Image Analysis
  Creases
Scalar Invariants in Multivariate Visualization

- Multivariate data ↔ scalar image analysis in computer vision

- Used in flow visualization:
  - Vortex core lines: valley lines of $\lambda_2$ (Sahner et al., '05)
  - Separating flow manifolds: ridge surfaces of FTLE
    (Haller et al., '07, Sadlo & Peikert, '07, Garth et al., '07)
Tensor Invariants

\[ \Theta = \cos^{-1}(\text{mode})/3 \]

\[ \lambda_1 = \frac{\text{tr}(D)}{3} + \sqrt{2/3} |E| \cos \left( \Theta - \frac{2\pi}{3} \right) \]

\[ \lambda_2 = \frac{\text{tr}(D)}{3} + \sqrt{2/3} |E| \cos \left( \Theta + \frac{2\pi}{3} \right) \]

\[ \lambda_3 = \frac{\text{tr}(D)}{3} + \sqrt{2/3} |E| \cos \left( \Theta + \frac{2\pi}{3} \right) \]

planar
mode = -1

orthotropic
mode = 0

linear
mode = 1
Two Invariants: FA and Mode
Our method: invariant crease lines

- **Salient tensor structures** as ridges and valleys of tensor invariants
- **Features in tensor field topology:** ridge and valley lines of tensor mode
- **DT-MRI data studied with** ridge and valley lines of FA
Outline

• Context and motivation
• Theory and related work
• Algorithm
• Results
Algorithm Overview

1) Tensor reconstruction
2) Crease point extraction on faces
3) Connectivity and tracking
Tensor Reconstruction

- Smooth convolution-based reconstruction of tensor field coefficients
- 4-sample support, piecewise quintic C3 2nd-order accurate kernel: [Möller et al. 1997]

\[ Q(x,y,z) = q(x)q(y)q(z) \]

\[ q(x) = \begin{cases} 
 0 & |x| > 2 \\
 0.1x^5 - 0.75x^4 - 2x^3 - 2x^2 + 0.8 & 1 < |x| < 2 \\
 -0.3x^5 + 0.75x^4 - x^2 + 0.7 & 0 < |x| < 1 
\end{cases} \]

- Reconstruction of tensor field derivatives
- Analytic derivatives of non-linear invariants
Crease Point Extraction

- Solve on 2D cross section:
  \[ \vec{g} \cdot \vec{e}_2 = \vec{g} \cdot \vec{e}_3 = 0 \Rightarrow \vec{g} \parallel \vec{e}_1 \]

- Parallel Vector Operator \( \text{(Peikert and Roth, '99)} \)

1) Solve numerically \( \vec{c} = \vec{g} \times \vec{e}_1 = \vec{0} \)

Newton: \( \vec{c}'(u + du) = \vec{0} = \vec{c}'(u) + \nabla \vec{c} du \)

\[
\frac{\partial \vec{c}}{\partial u} = \frac{\partial \vec{g}}{\partial u} \times \vec{e}_1 + \vec{g} \times \frac{\partial \vec{e}_1}{\partial u}
\]
Crease Point Extraction

- Solve on 2D cross section:
  \[ \vec{g} \cdot \vec{e}_2 = \vec{g} \cdot \vec{e}_3 = 0 \Rightarrow \vec{g} \parallel \vec{e}_1 \]

- Parallel Vector Operator (Peikert and Roth, '99)

  2) Can be solved analytically in linear case

\[ \begin{align*}
\vec{g} &= G(u, v, 1)^T, \quad \vec{e}_1 = E(u, v, 1)^T \\
\vec{g} \parallel \vec{e}_1 &: \exists \mu \in \mathbb{R}, (E^{-1}G)(u, v, 1)^T = \mu(u, v, 1)^T
\end{align*} \]
Crease Point Extraction

• Adaptive refinement approach
• invariants are highly nonlinear
• Adaptive subdivision until close to good bilinear approx. in quad
• Split final quad and apply PVO
Connectivity

In each voxel

Tracking for degenerate lines

• Integrate along major / minor eigenvector of Hessian of mode
Outline

- Context and motivation
- Theory
- Related Work
- Algorithm
- Results
Results: creases of mode

Stress tensor field (double point load)
Topology: lines of degeneracy

\[ \lambda_1 = \lambda_2 \]
\[ \lambda_2 = \lambda_3 \]

\[ D_3 = 0 \]
Zheng & Pang, '04

grayscale: mode

\[ D_3 = \frac{27}{2} \mu_2^3 (1 - \text{mode}(D)^2) \]
Results: creases for DTI?

Schultz et al.’07: Topology [creases of mode] not anatomically meaningful for DTI

If not creases of tensor mode, perhaps anisotropy?
Results: creases of FA

synthetic helix dataset

FA = fractional anisotropy
tractography
FA ridge line
Ridge Lines of FA in Brain

425 lines with length > 15mm

Sort based on average ridge strength

= Fornix, Cingulum, Noodles

1.5 x 1.5 x 1.5 mm
FA ridge lines

4 different datasets
Lower resolution
Same parameters
Reliably and automatically detecting cingulum bundles
Future: integrate with crease surfaces

2.0 x 2.0 x 2.5 mm
Conclusion

• Versatile approach to extract the structural properties of tensor fields

• Example of general trend: multivariate data → scalar image processing
  • Tensor mode → topology
  • FA → neuroanatomy

• Synthesis of visualization and automated analysis for quantitative studies
Acknowledgments

• Xiaoqiang Zheng and Alex Pang: data set
• Teem: http://teem.sf.net
• Funding: NIH/NCRR CIBC, P41-RR12553-08, NIH P41-RR13218 (NAC), R01-MH074794, U41-RR019703
• Reviewers for feedback
Thank You