Parameter Estimation of the Homodyned K Distribution Based on Signal to Noise Ratio

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Abstract—The Homodyned K (H-K) distribution is very important in Ultrasound (US) imaging because it models the speckle in a general way, even if the number of scatterers per resolution cell is low and if a coherent component exists. Due to the analytical complexity of this distribution, only a few works based on the H-K have been reported and other simpler distributions have been used instead. Here we provide evidence that, once we have the signal to noise ratio (SNR) from a set of parameters of the H-K distribution, we can estimate the parameters of any given sample set in an easy computational way. In addition, we present a validation study for our estimation procedure.

I. INTRODUCTION

Rayleigh [1], Rice [2], [3] and K [4], [5] distributions are widely used to model speckle in US images. However, they are not suited if both low scatterers density appears and a coherent component exists. All these effects are modeled by the Homodyned K (H-K) distribution, whose drawback is its analytical complexity. In order to avoid this, other authors [6], [7], used the Nakagami distribution [8] instead even though it is not clear whether this simpler distribution is equivalent to the H-K.

Some few works have been proposed to deal with the parameter estimation of the H-K distribution. In [9] the authors make use of the closed form expressions of even order moments 2, 4 and 6 of the H-K. However, in [10] an alternative estimation procedure based on lower order moments would be preferable. Even though a high activity on parameter estimation procedure based on lower order moments would make use of the closed form expressions of even order moments, in [10] an alternative estimation procedure based on lower order moments would be preferable. Even though a high activity on parameter estimation procedure based on lower order moments would be preferable.

In this work we demonstrate that, once we have the SNR from a set of parameters of the H-K, we can estimate the parameters of any given sample set in an easy way.

The paper is structured as follows: next section describe in brief the speckle model theory, introducing the probability distributions used to model speckle in US images, including the H-K distribution. In section III the SNR of the H-K is described and shown. Section IV explains our estimation method, which is validated in section V. Finally section VI presents our conclusions.

II. SPECKLE MODELS FOR ULTRASOUND ENVELOPES

A. Rayleigh, Rice and K Distributions

The resulting backscattered echo signal received in an US device can be expressed as the sum of the individual signals produced by the scatterers by

\[ Z = \sum_{i=0}^{N-1} x_i e^{i\phi_i} \] (1)

where \( \phi_i \) is the phase of every individual signal, and is a random variable uniformly distributed between the interval \((0, 2\pi)\), and is considered independent of \( x_i \). If the number of scatterers \( N \) in the resolution cell is large, then the envelope \( X \) of the signal is Rayleigh distributed [1], with a probability density function (PDF) given by

\[ f_R(x) = \frac{x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}}, \quad x > 0 \] (2)

where \( \sigma \) is the only parameter of interest, being \( 2\sigma^2 \) the total energy scattered through the receiver. If there exists a periodicity pattern in the scatterers location, or if there exists strong specular reflections, then a deterministic component appears. In this case the received signal can be expressed as

\[ Z = Z_s + \sum_{i=0}^{N-1} x_i e^{i\phi_i} \] (3)

where \( Z_s \) is the deterministic component of the signal. It can be shown that if the number of scatterers is large, the envelope of the received signal in this case can be modeled by a Rice distribution [2] as

\[ f_R(x) = \frac{x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} I_0 \left( \frac{sx}{\sigma^2} \right), \quad x > 0 \] (4)

where \( I_0(\cdot) \) is the modified Bessel function of first kind and zero order [12], and \( s \) is the envelope of the deterministic component. If the number of scatterers \( N \) is low\(^1\) and there is no deterministic component, the envelope follows a K-distribution [4] given by

\[ f_K(x) = 2 \left( \frac{x}{2} \right)^{\alpha} \frac{b^{\alpha+1}}{\Gamma(\alpha)} K_{\alpha-1}(bx), \quad x > 0 \] (5)

\(^1\)In this case \( N \) is considered as a random variable following a Gamma distribution.
where \( \Gamma(\cdot) \) is the Gamma function, \( K_\beta(\cdot) \) is the modified Bessel function of the second kind and order \( \beta \) \cite{12}, \( b = \sqrt{2\alpha/\sigma} \) and \( \alpha \) is the effective mean number of scatterers per resolution cell.

**B. Homodyned-K Distribution**

All the probability distributions mentioned above are valid only when some restrictions about the number of scatterers \( N \) or about the existence of a deterministic component \( Z_d \) are assumed. The more general PDF, modeling speckle envelopes without restrictions, is the Homodyned K distribution, whose PDF can be expressed as

\[
f_{HK}(x) = \frac{1}{\Gamma(\alpha)} \sqrt{\frac{2\alpha}{\pi\sigma^2}} \sum_{m=0}^{\infty} \left( -1 \right)^m \Gamma\left( \frac{\alpha}{2} + m \right) \left( \frac{\alpha^2}{s^2 \sigma^2} \right)^m \left( \frac{\alpha}{2\sigma^2} \right)^{\alpha+m-\frac{1}{2}} |x - s|^{\alpha+m-\frac{1}{2}} K_{\alpha+m-\frac{1}{2}} \left( |x - s| \sqrt{\frac{2\alpha}{\sigma^2}} \right)
\]

where \( \Gamma(\cdot) \) and \( K_\beta(\cdot) \), are the same as in the K distribution. The parameters of this distribution are: the effective mean number of scatterers \( \alpha \), the envelope \( s \) of the coherent signal \( Z_c \) and the mean energy of the non coherent scatterers given by \( 2\sigma^2 \). This PDF can be expressed in a simpler way as a function of the Rice distribution

\[
f_{HK}(x) = \int_0^\infty f_r(x/z) f_G(z) dz
\]

where \( f_r(x/z) = \frac{x^{\alpha}}{\sigma^2 z^2} \left( \frac{\alpha}{2\sigma^2} \right)^{\alpha+m-\frac{1}{2}} \left( \frac{x}{2\sigma^2} \right) I_0 \left( \frac{sx\alpha}{\sigma^2 z} \right), \quad x > 0
\]

and \( f_G(z) \) is the Gamma distribution with shape parameter \( \alpha \) and amplitude parameter unity \cite{13}. From equations (7) and (8) it is clear that one can obtain samples of the H-K distribution by creating independent and identically distributed (IID) samples of a Gamma random variable (with the parameters as mentioned above), say \( z_j, j = 1, ..., J \) and, for each \( z_j \), create samples of the appropriate Rice random variable. We have done so with \( J = 10^8 \).

In figure 1 we show plots of the PDF of the H-K, using different parameters of \( \alpha \), and a new parameter \( k = s/\sigma \), that can be referred to as coherent to diffuse signal ratio and that we will use in what follows. It can be shown that when the deterministic component tends to zero, the H-K distribution tends to the K distribution, when the number of scatterers tends to infinity the H-K distribution tends to the Rice distribution, when the number of scatterers tends to infinity and the deterministic component tends to zero the H-K distribution tends to the Rayleigh distribution, when both the deterministic component and the number of scatterers tends to infinity, the H-K tends to the Gaussian distribution, and when both the deterministic component and the number of scatterers tends to zero, the H-K tends to the log-normal distribution.

**III. SIGNAL TO NOISE RATIO**

When processing speckled imagery, the SNR concept is often used to evaluate the results of the algorithms applied. This parameter has been extended and defined as a ratio of moments of the variable\(^2\) \cite{10},

\[
\text{SNR}_v = \frac{m_v}{\sqrt{m_{2v} - m_v^2}} = \frac{E\{X^v\}}{\sqrt{E\{X^{2v}\} - E\{X^v\}^2}}
\]

being parameter \( v \) a non-negative constant. For \( v \) an even integer, closed form expressions of the moments have been reported \cite{14}. However, it has been pointed out in \cite{10}, that fractional low order moments of the K distribution are estimated from data with less variance than larger (integer) order moments, so they are preferable for SNR sampling computation. The authors in \cite{10} however, work with the K distribution, but not with the H-K. This gap is filled in this paper by showing curves of \( \text{SNR}_v \) for different values of \( v \) for the H-K distribution.

It deserves to be highlighted that the \( \text{SNR}_v \) from the H-K, for a given order moment \( v \), is a function of only two parameters, namely \( \alpha \) and \( k \), even though the H-K is a function of three parameters. This allows for a simple 2D representation (and also for a simpler estimation from data in a real application). In order to make the SNR parameter usable in real applications in which the H-K is involved, we have obtained a sample calculation of equation (9), using the procedure described in section II-B to generate IID samples of a H-K variable. We show in figures 2 and 3 plots of the \( \text{SNR}_v \), for two values of \( v \), as a function of \( \alpha \) and \( k \). Each contour in those figures shows the trajectory of pairs \( (\alpha, k) \) for any particular value of \( \text{SNR}_v \). To the best of our knowledge, these curves have never been reported, even though they constitute an important piece of information for parameter estimation in H-K data, as we show in the next section.

**IV. PARAMETER ESTIMATION USING THE SNR LEVEL CURVES**

We propose a parameter estimation of the H-K distribution using SNR level curves, which are obtained analytically using the closed form expressions for even order moments and numerically for fractional low order and odd order moments. They are shown in figures 2 and 3 for order moments \( v = 2 \)

\(^2\)We will refer to the concept signal to noise ratio either by SNR or by \( \text{SNR}_v \). In the latter case we intend to emphasize the dependence of this parameter with the baseline order moment \( v \).
and $v = 0.25$. The simplest version of our procedure is to estimate two values of SNR$_v$ for the sample set and then to cross the level curves for these two SNR values. The crossing-point gives the estimated $\alpha$ and $k$ values (see figures 4 and 5). For small sample sizes, it may be more appropriate to overlap several curves and to find the pair $(\alpha,k)$ that minimizes some error criterion. The approach we propose here is to use several SNR curves. We have used those with $v = \{0.01, 0.03, 0.05, 0.075, 0.1, 0.25, 0.4, 0.5, 0.75, 1\}$. Using these curves, we compute at each pixel of a 2D grid, the number of curves that pass through it. The pixel with a maximum value will be located at the intersection of the maximum number of curves, and it will give us the estimated value for $k$ and $\alpha$. Figure 6 shows this procedure, where warm colors correspond with high values and cold colors with low values. This method is sensitive to the resolution selected for the 2D grid. Choosing a high resolution grid will arise to a more accurate estimation, but it will not assure that a global maximum could be located, because only a few curves will meet at the same pixel. On the other hand, choosing a low resolution grid assures that a global maximum could be easily located, but at the cost of weak estimation. For these reasons a multiresolution approach could be employed to refine the values starting from a low resolution grid and making it finer until the global maximum is located or the precision needed is achieved. We have used a grid of size $250 \times 250$, but we have also obtained good results using grid sizes up to $1000 \times 1000$ starting from a minimum of $100 \times 100$, for parameters ranges of $k \in [0, 10]$, and $\alpha \in [0.1, 100]$.

V. VALIDATION STUDY

In order to validate our estimation method we have carried out an experiment using several sample sizes in the estimation procedure. In figures 7 and 8 we show the standard deviation (STD) and mean values obtained for the estimation of parameters $k$ and $\alpha$ using several sample sizes from 50 up to $10^5$, and repeating this experiment 25 times in order to estimate mean and STD. It is clear from these figures that the estimation carried out converges to a stable value when the number of samples is big enough, and its STD tends to zero. $k$ is correctly estimated using 1000 or more samples, and $\alpha$ is correctly estimated using sample sizes of 2500 or more. The estimation of $k$ is better than $\alpha$, because the SNR level
curves tends to be more horizontal than vertical, therefore the resolution is better resolved in the vertical axis (the “k” axis).

VI. CONCLUSIONS

This work is, to our knowledge, one of the few ones dealing with parameter estimation of the H-K distribution using the SNR, and in addition, one of the few that make use of the fractional low order moments of the H-K distribution. The methodology described here to estimate the parameters $k$ and $\alpha$, starts obtaining first the SNR numerically by means of IID sampling, using fractional low order moments. We have shown that the data obtained from that simulation can be represented by means of SNR level curves (see figures 4 and 5) as a function of the parameters, providing important information about the H-K distribution. Using these curves we have proposed a novel method that allows the estimation of $k$ and $\alpha$ using a simple computational method, as shown in figure 6, which have been proved valid in section V, as the estimation of the parameters improves when the sample size increases. As far as we know, this is the first time this methodology is used, and therefore this would open up new paths for the study of the complex H-K distribution, which has been shown very important for US speckle modeling.

We have not made explicit mention of the controversy about using this complicated distribution or to use simpler approximated distributions, such as the Nakagami. This is, in our opinion, a different issue which falls beyond the scope of this paper.

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